

# Analogs of Cramer's rule for the least squares solutions of some matrix equations.

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## Abstract

The least squares solutions with the minimum norm of the matrix equations  $\mathbf{AX} = \mathbf{B}$ ,  $\mathbf{XA} = \mathbf{B}$  and  $\mathbf{AXB} = \mathbf{D}$  are considered in this paper. We use the determinantal representations of the Moore - Penrose inverse obtained earlier by the author and get analogs of the Cramer rule for the least squares solutions of these matrix equations.

*Keywords:* Moore-Penrose inverse, matrix equation, least squares solution, Cramer rule

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## 1 Introduction

In this paper we shall adopt the following notation. Let  $\mathbb{C}^{m \times n}$  be the set of  $m$  by  $n$  matrices with complex entries,  $\mathbb{C}_r^{m \times n}$  be a subset of  $\mathbb{C}^{m \times n}$  in which any matrix has rank  $r$ ,  $\mathbf{I}_m$  be the identity matrix of order  $m$ , and  $\|\cdot\|$  be the Frobenius norm of a matrix.

Denote by  $\mathbf{a}_j$  and  $\mathbf{a}_i$  the  $j$ th column and the  $i$ th row of  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , respectively. Then  $\mathbf{a}_j^*$  and  $\mathbf{a}_i^*$  denote the  $j$ th column and the  $i$ th row of a Hermitian adjoint matrix  $\mathbf{A}^*$  as well. Let  $\mathbf{A}_{\cdot j}(\mathbf{b})$  denote the matrix obtained from  $\mathbf{A}$  by replacing its  $j$ th column with the vector  $\mathbf{b}$ , and by  $\mathbf{A}_i(\mathbf{b})$  denote the matrix obtained from  $\mathbf{A}$  by replacing its  $i$ th row with  $\mathbf{b}$ .

Let  $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$  and  $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$  be subsets of the order  $1 \leq k \leq \min\{m, n\}$ . Then  $|\mathbf{A}_{\beta}^{\alpha}|$  denotes the minor

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of  $\mathbf{A}$  determined by the rows indexed by  $\alpha$  and the columns indexed by  $\beta$ . Clearly,  $|\mathbf{A}_\alpha^\alpha|$  be a principal minor determined by the rows and columns indexed by  $\alpha$ . For  $1 \leq k \leq n$ , denote by

$$L_{k,n} := \{ \alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 \leq \dots \leq \alpha_k \leq n \}$$

the collection of strictly increasing sequences of  $k$  integers chosen from the set  $\{1, \dots, n\}$ . For fixed  $i \in \alpha$  and  $j \in \beta$ , let

$$I_{r,m}\{i\} := \{ \alpha : \alpha \in L_{r,m}, i \in \alpha \}, \quad J_{r,n}\{j\} := \{ \beta : \beta \in L_{r,n}, j \in \beta \}.$$

Matrix equation is one of the important study fields of linear algebra. Linear matrix equations, such as

$$\mathbf{AX} = \mathbf{B}, \tag{1}$$

$$\mathbf{XB} = \mathbf{D}, \tag{2}$$

and

$$\mathbf{AXB} = \mathbf{D}, \tag{3}$$

play an important role in linear system theory therefore a large number of papers have presented several methods for solving these matrix equations [1, 2, 3, 4, 5]. In [6], Khatri and Mitra studied the Hermitian solutions to the matrix equations (1) and (3) over the complex field and the system of the equations (1) and (2). Wang, in [7, 8], and Li and Wu, in [9] studied the bisymmetric, symmetric and skew-antisymmetric least squares solution to this system over the quaternion skew field. Extreme ranks of real matrices in least squares solution of the equation (3) was investigated in [10] over the complex field and in [11] over the quaternion skew field.

As we know, the Cramer rule gives an explicit expression for the solution of nonsingular linear equations. In [12], Robinson gave its elegant proof over the complex field which aroused great interest in finding determinantal formulas as analogs of the Cramer rule for the matrix equations [13, 14, 15, 16, 17, 18, 20, 22, 23, 24]. The Cramer rule for solutions of the restricted matrix equations (1), (2) and (3) was established in [21].

In this paper, we use the results of [16] to obtain the Cramer rule for least squares solutions of the matrix equations (1), (2) and (3) without any restriction. The paper is organized as follows. We start with some basic concepts and results about determinantal representations of the Moore - Penrose inverse in Section 2. In Section 3, we derive some generalized Cramer rules for the matrix equations (1), (2) and (3). In Section 4, we show a numerical example to illustrate the main result.

## 2 Determinantal representations of the Moore - Penrose inverse

**Definition 2.1** If  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , then the matrix  $\mathbf{A}^+$  is called the Moore - Penrose inverse of  $\mathbf{A}$  if it satisfies the equations: 1)  $(\mathbf{A}\mathbf{A}^+)^* = \mathbf{A}\mathbf{A}^+$ ; 2)  $(\mathbf{A}^+\mathbf{A})^* = \mathbf{A}^+\mathbf{A}$ ; 3)  $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$ ; 4)  $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$ .

It is well known the following proposition.

**Lemma 2.1** For an arbitrary  $\mathbf{A} \in \mathbb{C}^{m \times n}$  there exists a unique Moore - Penrose inverse  $\mathbf{A}^+$ .

**Lemma 2.2** [25] If  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , then

$$\mathbf{A}^+ = \lim_{\lambda \rightarrow 0} \mathbf{A}^* (\mathbf{A}\mathbf{A}^* + \lambda \mathbf{I})^{-1} = \lim_{\lambda \rightarrow 0} (\mathbf{A}^* \mathbf{A} + \lambda \mathbf{I})^{-1} \mathbf{A}^*,$$

where  $\lambda \in \mathbb{R}_+$ , and  $\mathbb{R}_+$  is a set of the real positive numbers.

**Corollary 2.1** If  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , then the following statements are true.

- i) If  $\text{rank } \mathbf{A} = n$ , then  $\mathbf{A}^+ = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^*$ .
- ii) If  $\text{rank } \mathbf{A} = m$ , then  $\mathbf{A}^+ = \mathbf{A}^* (\mathbf{A}\mathbf{A}^*)^{-1}$ .
- iii) If  $\text{rank } \mathbf{A} = n = m$ , then  $\mathbf{A}^+ = \mathbf{A}^{-1}$ .

**Theorem 2.1** [16] If  $\mathbf{A} \in \mathbb{C}_r^{m \times n}$  and  $r < \min\{m, n\}$ , then the Moore-Penrose inverse  $\mathbf{A}^+ = (a_{ij}^+) \in \mathbb{C}^{n \times m}$  possess the following determinantal representations:

$$a_{ij}^+ = \frac{\sum_{\beta \in J_{r, n} \setminus \{i\}} \left| ((\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot j}^*))_{\beta} \right|}{\sum_{\beta \in J_{r, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta} \right|}, \quad (4)$$

or

$$a_{ij}^+ = \frac{\sum_{\alpha \in I_{r, m} \setminus \{j\}} \left| ((\mathbf{A}\mathbf{A}^*)_{j \cdot} (\mathbf{a}_{i \cdot}^*))_{\alpha} \right|}{\sum_{\alpha \in I_{r, m}} \left| (\mathbf{A}\mathbf{A}^*)_{\alpha} \right|}. \quad (5)$$

**Remark 2.1** If  $\text{rank } \mathbf{A} = n$ , then by Corollary 2.1  $\mathbf{A}^+ = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^*$ . Therefore, we get the following representation of  $\mathbf{A}^+$ :

$$\mathbf{A}^+ = \frac{1}{\det(\mathbf{A}^* \mathbf{A})} \begin{pmatrix} \det(\mathbf{A}^* \mathbf{A})_{.1}(\mathbf{a}_{.1}^*) & \dots & \det(\mathbf{A}^* \mathbf{A})_{.1}(\mathbf{a}_{.m}^*) \\ \dots & \dots & \dots \\ \det(\mathbf{A}^* \mathbf{A})_{.n}(\mathbf{a}_{.1}^*) & \dots & \det(\mathbf{A}^* \mathbf{A})_{.n}(\mathbf{a}_{.m}^*) \end{pmatrix} \quad (6)$$

If  $\text{rank } \mathbf{A} = n < m$ , then by Theorem 2.1 for  $\mathbf{A}^+$  we have (4) as well.

**Remark 2.2** If  $\text{rank } \mathbf{A} = m$ , then by Corollary 2.1  $\mathbf{A}^+ = \mathbf{A}^* (\mathbf{A} \mathbf{A}^*)^{-1}$ . Hence, we obtain the following representation of  $\mathbf{A}^+$ :

$$\mathbf{A}^+ = \frac{1}{\det(\mathbf{A} \mathbf{A}^*)} \begin{pmatrix} \det(\mathbf{A} \mathbf{A}^*)_{1.}(\mathbf{a}_{1.}^*) & \dots & \det(\mathbf{A} \mathbf{A}^*)_{m.}(\mathbf{a}_{1.}^*) \\ \dots & \dots & \dots \\ \det(\mathbf{A} \mathbf{A}^*)_{1.}(\mathbf{a}_{n.}^*) & \dots & \det(\mathbf{A} \mathbf{A}^*)_{m.}(\mathbf{a}_{n.}^*) \end{pmatrix} \quad (7)$$

If  $\text{rank } \mathbf{A} = m < n$ , then by Theorem 2.1 for  $\mathbf{A}^+$  we also have (5).

### 3 Cramer's rule of the least squares solution of some matrix equation

**Definition 3.1** Consider a matrix equation (1), where  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{C}^{m \times s}$  are given,  $\mathbf{X} \in \mathbb{C}^{n \times s}$  is unknown. Suppose

$$S_1 = \{\mathbf{X} | \mathbf{X} \in \mathbb{C}^{n \times s}, \|\mathbf{A}\mathbf{X} - \mathbf{B}\| = \min\}.$$

Then matrices  $\mathbf{X} \in \mathbb{C}^{n \times s}$  such that  $\mathbf{X} \in S_1$  are called least squares solutions of the matrix equation (1). If  $\mathbf{X}_{LS} = \min_{\mathbf{X} \in S_1} \|\mathbf{X}\|$ , then  $\mathbf{X}_{LS}$  is called the least squares solution of (1) with minimum norm.

If the equation (1) has no precision solutions, then  $\mathbf{X}_{LS}$  is its maximum approximate solution.

The following important theorem is well-known.

**Theorem 3.1** ([15]) The least squares solutions of (1) are

$$\mathbf{X} = \mathbf{A}^+ \mathbf{B} + (\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}) \mathbf{C},$$

in which  $\mathbf{C} \in \mathbb{C}^{n \times s}$  is an arbitrary matrix and the minimum norm solution is  $\mathbf{X}_{LS} = \mathbf{A}^+ \mathbf{B}$ .

We denote  $\mathbf{A}^* \mathbf{B} =: \hat{\mathbf{B}} = (\hat{b}_{ij}) \in \mathbb{C}^{n \times s}$ .

**Theorem 3.2** (i) If  $\text{rank } \mathbf{A} = n$ , then for the least squares solution  $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times s}$  for all  $i = \overline{1, n}$ ,  $j = \overline{1, s}$  we have

$$x_{ij} = \frac{\det(\mathbf{A}^* \mathbf{A})_{.i} (\hat{\mathbf{b}}_{.j})}{\det(\mathbf{A}^* \mathbf{A})} \quad (8)$$

where  $\hat{\mathbf{b}}_{.j}$  is the  $j$ th column of  $\hat{\mathbf{B}}$  for all  $j = \overline{1, s}$ .

(ii) If  $\text{rank } \mathbf{A} = r \leq m < n$ , then for all  $i = \overline{1, n}$ ,  $j = \overline{1, s}$  we have

$$x_{ij} = \frac{\sum_{\beta \in J_{r, n} \{j\}} \left| \left( (\mathbf{A}^* \mathbf{A})_{.i} (\hat{\mathbf{b}}_{.j}) \right)_{\beta} \right|}{\sum_{\beta \in J_{r, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta} \right|}. \quad (9)$$

*Proof.* i) If  $\text{rank } \mathbf{A} = n$ , then by Corollary 2.1  $\mathbf{A}^+ = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^*$ . By multiplying  $\mathbf{A}^+ \mathbf{B} = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^* \mathbf{B} = (\mathbf{A}^* \mathbf{A})^{-1} \hat{\mathbf{B}}$ , we obtain for all  $i = \overline{1, n}$ ,  $j = \overline{1, s}$

$$x_{ij} = \frac{1}{\det(\mathbf{A}^* \mathbf{A})} \sum_{k=1}^n L_{ki} \hat{b}_{kj},$$

where  $L_{ij}$  is a  $ij$ th cofactor of  $(\mathbf{A}^* \mathbf{A})$  for all  $i, j = \overline{1, n}$ . Denoting the  $j$ th column of  $\hat{\mathbf{B}}$  by  $\hat{\mathbf{b}}_{.j}$ , it follows (8).

ii) If  $\text{rank } \mathbf{A} = r \leq m < n$ , then by Theorem 2.1 we can represent the matrix  $\mathbf{A}^+$  by (4). Therefore, we obtain for all  $i = \overline{1, n}$ ,  $j = \overline{1, s}$

$$\begin{aligned} x_{ij} &= \sum_{k=1}^m a_{ik}^+ b_{kj} = \sum_{k=1}^m \frac{\sum_{\beta \in J_{r, n} \{i\}} \left| \left( (\mathbf{A}^* \mathbf{A})_{.i} (\mathbf{a}_{.k}^*) \right)_{\beta} \right|}{\sum_{\beta \in J_{r, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta} \right|} \cdot b_{kj} = \\ &= \frac{\sum_{\beta \in J_{r, n} \{i\}} \sum_{k=1}^m \left| \left( (\mathbf{A}^* \mathbf{A})_{.i} (\mathbf{a}_{.k}^*) \right)_{\beta} \right| \cdot b_{kj}}{\sum_{\beta \in J_{r, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta} \right|}. \end{aligned}$$

Since  $\sum_k \mathbf{a}_{\cdot k}^* b_{kj} = \begin{pmatrix} \sum_k a_{1k}^* b_{kj} \\ \sum_k a_{2k}^* b_{kj} \\ \vdots \\ \sum_k a_{nk}^* b_{kj} \end{pmatrix} = \hat{\mathbf{b}}_{\cdot j}$ , then it follows (9). ■

**Definition 3.2** Consider a matrix equation

$$\mathbf{X}\mathbf{A} = \mathbf{B}, \quad (10)$$

where  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{C}^{s \times n}$  are given,  $\mathbf{X} \in \mathbb{C}^{s \times m}$  is unknown. Suppose

$$S_2 = \{\mathbf{X} | \mathbf{X} \in \mathbb{C}^{s \times m}, \|\mathbf{X}\mathbf{A} - \mathbf{B}\| = \min\}.$$

Then matrices  $\mathbf{X} \in \mathbb{C}^{s \times m}$  such that  $\mathbf{X} \in S_2$  are called least squares solutions of the matrix equation (10). If  $\mathbf{X}_{LS} = \min_{\mathbf{X} \in S_2} \|\mathbf{X}\|$ , then  $\mathbf{X}_{LS}$  is called the least squares solution of (10) with minimum norm.

The following theorem can be obtained by analogy to Theorem 3.1.

**Theorem 3.3** The least squares solutions of (10) are

$$\mathbf{X} = \mathbf{B}\mathbf{A}^+ + \mathbf{C}(\mathbf{I}_m - \mathbf{A}\mathbf{A}^+),$$

in which  $\mathbf{C} \in \mathbb{C}^{s \times m}$  is an arbitrary matrix and the minimum norm solution is  $\mathbf{X}_{LS} = \mathbf{B}\mathbf{A}^+$ .

We denote  $\mathbf{B}\mathbf{A}^* =: \check{\mathbf{B}} = (\check{b}_{ij}) \in \mathbb{C}^{s \times m}$ .

**Theorem 3.4** (i) If  $\text{rank } \mathbf{A} = m$ , then for the least squares solution  $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{s \times m}$  for all  $i = \overline{1, s}$ ,  $j = \overline{1, m}$  we have

$$x_{ij} = \frac{\det(\mathbf{A}\mathbf{A}^*)_{j \cdot} (\check{\mathbf{b}}_{i \cdot})}{\det(\mathbf{A}\mathbf{A}^*)} \quad (11)$$

where  $\check{\mathbf{b}}_{i \cdot}$  is the  $i$ th row of  $\check{\mathbf{B}}$  for all  $i = \overline{1, s}$ .

(ii) If  $\text{rank } \mathbf{A} = r \leq n < m$ , then for all  $i = \overline{1, s}$ ,  $j = \overline{1, m}$  we have

$$x_{ij} = \frac{\sum_{\alpha \in I_{r, m} \setminus \{i\}} \left| \left( (\mathbf{A}\mathbf{A}^*)_{j \cdot} (\check{\mathbf{b}}_{i \cdot}) \right)_{\alpha} \right|}{\sum_{\alpha \in I_{r, m}} \left| (\mathbf{A}\mathbf{A}^*)_{\alpha} \right|}. \quad (12)$$

*Proof.* i) If  $\text{rank } \mathbf{A} = m$ , then by Corollary 2.1  $\mathbf{A}^+ = \mathbf{A}^* (\mathbf{A} \mathbf{A}^*)^{-1}$ . By multiplying  $\mathbf{B} \mathbf{A}^+ = \mathbf{B} \mathbf{A}^* (\mathbf{A} \mathbf{A}^*)^{-1} = \check{\mathbf{B}} (\mathbf{A} \mathbf{A}^*)^{-1}$ , we obtain for all  $i = \overline{1, s}$ ,  $j = \overline{1, m}$ ,

$$x_{ij} = \frac{1}{\det(\mathbf{A} \mathbf{A}^*)} \sum_{k=1}^m \check{b}_{ik} R_{jk},$$

where  $R_{ij}$  is a  $ij$ th cofactor of  $(\mathbf{A} \mathbf{A}^*)$  for all  $i, j = \overline{1, m}$ . Denoting the  $i$ th row of  $\check{\mathbf{B}}$  by  $\check{\mathbf{b}}_i$ , it follows (11).

ii) If  $\text{rank } \mathbf{A} = r \leq n < m$ , then by Theorem 2.1 we can represent the matrix  $\mathbf{A}^+$  by (5). Therefore, for all  $i = \overline{1, s}$ ,  $j = \overline{1, m}$  we obtain

$$\begin{aligned} x_{ij} &= \sum_{k=1}^n b_{ik} a_{kj}^+ = \sum_{k=1}^n b_{ik} \cdot \frac{\sum_{\alpha \in I_{r,m}\{i\}} \left| \left( (\mathbf{A} \mathbf{A}^*)_{j \cdot} (\mathbf{a}_k^*) \right)_{\alpha} \right|}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A} \mathbf{A}^*)_{\alpha}|} = \\ &= \frac{\sum_{k=1}^n b_{ik} \sum_{\alpha \in I_{r,m}\{i\}} \left| \left( (\mathbf{A} \mathbf{A}^*)_{j \cdot} (\mathbf{a}_k^*) \right)_{\alpha} \right|}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A} \mathbf{A}^*)_{\alpha}|} \end{aligned}$$

Since for all  $i = \overline{1, s}$

$$\sum_k b_{ik} \mathbf{a}_k^* = \begin{pmatrix} \sum_k b_{ik} a_{k1}^* & \sum_k b_{ik} a_{k2}^* & \cdots & \sum_k b_{ik} a_{km}^* \end{pmatrix} = \check{\mathbf{b}}_i,$$

then it follows (12). ■

**Definition 3.3** Consider a matrix equation (3), where  $\mathbf{A} \in \mathbb{C}_{r_1}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{C}_{r_2}^{p \times q}$ ,  $\mathbf{D} \in \mathbb{C}^{m \times q}$  are given,  $\mathbf{X} \in \mathbb{C}^{n \times p}$  is unknown. Suppose

$$S_3 = \{\mathbf{X} \mid \mathbf{X} \in \mathbb{C}^{n \times p}, \|\mathbf{A} \mathbf{X} \mathbf{B} - \mathbf{D}\| = \min\}.$$

Then matrices  $\mathbf{X} \in \mathbb{C}^{n \times p}$  such that  $\mathbf{X} \in S_3$  are called least squares solutions of the matrix equation (3). If  $\mathbf{X}_{LS} = \min_{\mathbf{X} \in S_3} \|\mathbf{X}\|$ , then  $\mathbf{X}_{LS}$  is called the least squares solution of (3) with minimum norm.

The following important theorem is well-known.

**Theorem 3.5** ([24]) The least squares solutions of (3) are

$$\mathbf{X} = \mathbf{A}^+ \mathbf{D} \mathbf{B}^+ + (\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}) \mathbf{V} + \mathbf{W} (\mathbf{I}_p - \mathbf{B} \mathbf{B}^+),$$

in which  $\{\mathbf{V}, \mathbf{W}\} \subset \mathbb{C}^{n \times p}$  are arbitrary quaternion matrices and the least squares solution with minimum norm is  $\mathbf{X}_{LS} = \mathbf{A}^+ \mathbf{D} \mathbf{B}^+$ .

We denote  $\tilde{\mathbf{D}} = \mathbf{A}^* \mathbf{D} \mathbf{B}^*$ .

**Theorem 3.6** (i) If  $\text{rank } \mathbf{A} = n$  and  $\text{rank } \mathbf{B} = p$ , then for the least squares solution  $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$  of (3) we have for all  $i = \overline{1, n}$ ,  $j = \overline{1, p}$ ,

$$x_{ij} = \frac{\det((\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{d}_{\cdot j}^{\mathbf{B}}))}{\det(\mathbf{A}^* \mathbf{A}) \cdot \det(\mathbf{B} \mathbf{B}^*)}, \quad (13)$$

or

$$x_{ij} = \frac{\det((\mathbf{B} \mathbf{B}^*)_{j \cdot} (\mathbf{d}_{i \cdot}^{\mathbf{A}}))}{\det(\mathbf{A}^* \mathbf{A}) \cdot \det(\mathbf{B} \mathbf{B}^*)}, \quad (14)$$

where

$$\mathbf{d}_{\cdot j}^{\mathbf{B}} := \left[ \det((\mathbf{B} \mathbf{B}^*)_{j \cdot} (\tilde{\mathbf{d}}_{1 \cdot})) , \dots , \det((\mathbf{B} \mathbf{B}^*)_{j \cdot} (\tilde{\mathbf{d}}_{n \cdot})) \right]^T, \quad (15)$$

$$\mathbf{d}_{i \cdot}^{\mathbf{A}} := \left[ \det((\mathbf{A}^* \mathbf{A})_{\cdot i} (\tilde{\mathbf{d}}_{1 \cdot})) , \dots , \det((\mathbf{A}^* \mathbf{A})_{\cdot i} (\tilde{\mathbf{d}}_{p \cdot})) \right] \quad (16)$$

are respectively the column-vector and the row-vector.  $\tilde{\mathbf{d}}_{i \cdot}$  is the  $i$ th row of  $\tilde{\mathbf{D}}$  for all  $i = \overline{1, n}$ , and  $\tilde{\mathbf{d}}_{\cdot j}$  is the  $j$ th column of  $\tilde{\mathbf{D}}$  for all  $j = \overline{1, p}$ .

(ii) If  $\text{rank } \mathbf{A} = r_1 < m$  and  $\text{rank } \mathbf{B} = r_2 < p$ , then for the least squares solution  $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$  of (3) we have

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{d}_{\cdot j}^{\mathbf{B}})_{\beta}^{\beta} \right|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right| \sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|}, \quad (17)$$

or

$$x_{ij} = \frac{\sum_{\alpha \in I_{r_2, p}\{j\}} \left| (\mathbf{B} \mathbf{B}^*)_{j \cdot} (\mathbf{d}_{i \cdot}^{\mathbf{A}})_{\alpha}^{\alpha} \right|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right| \sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|}, \quad (18)$$

where

$$\mathbf{d}_{\cdot j}^{\mathbf{B}} = \left[ \sum_{\alpha \in I_{r_2, p}\{j\}} \left| (\mathbf{B} \mathbf{B}^*)_{j \cdot} (\tilde{\mathbf{d}}_{1 \cdot})_{\alpha}^{\alpha} \right| , \dots , \sum_{\alpha \in I_{r_2, p}\{j\}} \left| (\mathbf{B} \mathbf{B}^*)_{j \cdot} (\tilde{\mathbf{d}}_{n \cdot})_{\alpha}^{\alpha} \right| \right]^T, \quad (19)$$



$$\mathbf{d}_{i.}^{\mathbf{A}} = \left[ \sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{.i} \left( \tilde{\mathbf{d}}_{.1} \right)_{\beta}^{\beta} \right|, \dots, \sum_{\alpha \in I_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{.i} \left( \tilde{\mathbf{d}}_{.p} \right)_{\beta}^{\beta} \right| \right] \quad (20)$$

are the column-vector and the row-vector, respectively.

(iii) If  $\text{rank } \mathbf{A} = n$  and  $\text{rank } \mathbf{B} = r_2 < p$ , then for the least squares solution  $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$  of (3) we have

$$x_{ij} = \frac{\det \left( (\mathbf{A}^* \mathbf{A})_{.i} \left( \mathbf{d}_{.j}^{\mathbf{B}} \right) \right)}{\det(\mathbf{A}^* \mathbf{A}) \sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|}, \quad (21)$$

or

$$\frac{\sum_{\alpha \in I_{r_2, p}\{j\}} \left| (\mathbf{B} \mathbf{B}^*)_{j.} \left( \mathbf{d}_{i.}^{\mathbf{A}} \right)_{\alpha}^{\alpha} \right|}{\det(\mathbf{A}^* \mathbf{A}) \sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|}, \quad (22)$$

where  $\mathbf{d}_{.j}^{\mathbf{B}}$  is (19) and  $\mathbf{d}_{i.}^{\mathbf{A}}$  is (16).

(iii) If  $\text{rank } \mathbf{A} = r_1 < m$  and  $\text{rank } \mathbf{B} = p$ , then for the least squares solution  $\mathbf{X}_{LS} = (x_{ij}) \in \mathbb{C}^{n \times p}$  of (3) we have

$$x_{ij} = \frac{\det \left( (\mathbf{B} \mathbf{B}^*)_{j.} \left( \mathbf{d}_{i.}^{\mathbf{A}} \right) \right)}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right| \cdot \det(\mathbf{B} \mathbf{B}^*)}, \quad (23)$$

or

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{.i} \left( \mathbf{d}_{.j}^{\mathbf{B}} \right)_{\beta}^{\beta} \right|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right| \det(\mathbf{B} \mathbf{B}^*)}, \quad (24)$$

where  $\mathbf{d}_{.j}^{\mathbf{B}}$  is (15) and  $\mathbf{d}_{i.}^{\mathbf{A}}$  is (20).

*Proof.* i) If  $\text{rank } \mathbf{A} = n$  and  $\text{rank } \mathbf{B} = p$ , then by Corollary 2.1  $\mathbf{A}^+ =$

$(\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^*$  and  $\mathbf{B}^+ = \mathbf{B}^* (\mathbf{B} \mathbf{B}^*)^{-1}$ . Therefore, we obtain

$$\begin{aligned} \mathbf{X}_{LS} &= (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^* \mathbf{D} \mathbf{B}^* (\mathbf{B} \mathbf{B}^*)^{-1} = \\ &= \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} = \frac{1}{\det(\mathbf{A}^* \mathbf{A})} \begin{pmatrix} L_{11}^{\mathbf{A}} & L_{21}^{\mathbf{A}} & \dots & L_{n1}^{\mathbf{A}} \\ L_{12}^{\mathbf{A}} & L_{22}^{\mathbf{A}} & \dots & L_{n2}^{\mathbf{A}} \\ \dots & \dots & \dots & \dots \\ L_{1n}^{\mathbf{A}} & L_{2n}^{\mathbf{A}} & \dots & L_{nn}^{\mathbf{A}} \end{pmatrix} \times \\ &\times \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \dots & \tilde{d}_{1m} \\ \tilde{d}_{21} & \tilde{d}_{22} & \dots & \tilde{d}_{2m} \\ \dots & \dots & \dots & \dots \\ \tilde{d}_{n1} & \tilde{d}_{n2} & \dots & \tilde{d}_{nm} \end{pmatrix} \frac{1}{\det(\mathbf{B} \mathbf{B}^*)} \begin{pmatrix} R_{11}^{\mathbf{B}} & R_{21}^{\mathbf{B}} & \dots & R_{p1}^{\mathbf{B}} \\ R_{12}^{\mathbf{B}} & R_{22}^{\mathbf{B}} & \dots & R_{p2}^{\mathbf{B}} \\ \dots & \dots & \dots & \dots \\ R_{1p}^{\mathbf{B}} & R_{2p}^{\mathbf{B}} & \dots & R_{pp}^{\mathbf{B}} \end{pmatrix}, \end{aligned}$$

where  $\tilde{d}_{ij}$  is  $ij$ th entry of the matrix  $\tilde{\mathbf{D}}$ ,  $L_{ij}^{\mathbf{A}}$  is the  $ij$ th cofactor of  $(\mathbf{A}^* \mathbf{A})$  for all  $i, j = \overline{1, n}$  and  $R_{ij}^{\mathbf{B}}$  is the  $ij$ th cofactor of  $(\mathbf{B} \mathbf{B}^*)$  for all  $i, j = \overline{1, p}$ . This implies

$$x_{ij} = \frac{\sum_{k=1}^n L_{ki}^{\mathbf{A}} \left( \sum_{s=1}^p \tilde{d}_{ks} R_{js}^{\mathbf{B}} \right)}{\det(\mathbf{A}^* \mathbf{A}) \cdot \det(\mathbf{B} \mathbf{B}^*)}, \quad (25)$$

for all  $i = \overline{1, n}$ ,  $j = \overline{1, p}$ . We obtain the sum in parentheses and denote it as follows

$$\sum_{s=1}^p \tilde{d}_{ks} R_{js}^{\mathbf{B}} = \det(\mathbf{B} \mathbf{B}^*)_{j.} \left( \tilde{\mathbf{d}}_{k.} \right) := d_{kj}^{\mathbf{B}},$$

where  $\tilde{\mathbf{d}}_{k.}$  is the  $k$ th row-vector of  $\tilde{\mathbf{D}}$  for all  $k = \overline{1, n}$ . Suppose  $\mathbf{d}_{.j}^{\mathbf{B}} := (d_{1j}^{\mathbf{B}}, \dots, d_{nj}^{\mathbf{B}})^T$  is the column-vector for all  $j = \overline{1, p}$ . Reducing the sum  $\sum_{k=1}^n L_{ki}^{\mathbf{A}} d_{kj}^{\mathbf{B}}$ , we obtain an analog of Cramer's rule for (3) by (13).

Interchanging the order of summation in (25), we have

$$x_{ij} = \frac{\sum_{s=1}^p \left( \sum_{k=1}^n L_{ki}^{\mathbf{A}} \tilde{d}_{ks} \right) R_{js}^{\mathbf{B}}}{\det(\mathbf{A}^* \mathbf{A}) \cdot \det(\mathbf{B} \mathbf{B}^*)}.$$

We obtain the sum in parentheses and denote it as follows

$$\sum_{k=1}^n L_{ki}^{\mathbf{A}} \tilde{d}_{ks} = \det(\mathbf{A}^* \mathbf{A})_{.i} \left( \tilde{\mathbf{d}}_{.s} \right) =: d_{is}^{\mathbf{A}},$$

where  $\tilde{\mathbf{d}}_{\cdot s}$  is the  $s$ th column-vector of  $\tilde{\mathbf{D}}$  for all  $s = \overline{1, p}$ . Suppose  $\mathbf{d}_{i\cdot}^{\mathbf{A}} := (d_{i1}^{\mathbf{A}}, \dots, d_{ip}^{\mathbf{A}})$  is the row-vector for all  $i = \overline{1, n}$ . Reducing the sum  $\sum_{s=1}^n d_{is}^{\mathbf{A}} R_{js}^{\mathbf{B}}$ , we obtain another analog of Cramer's rule for the least squares solutions of (3) by (14).

ii) If  $\mathbf{A} \in \mathbb{C}_{r_1}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{C}_{r_2}^{p \times q}$  and  $r_1 < n$ ,  $r_2 < p$ , then by Theorem 2.1 the Moore-Penrose inverses  $\mathbf{A}^+ = (a_{ij}^+) \in \mathbb{C}^{n \times m}$  and  $\mathbf{B}^+ = (b_{ij}^+) \in \mathbb{C}^{q \times p}$  possess the following determinantal representations respectively,

$$\begin{aligned} a_{ij}^+ &= \frac{\sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot j}^*)_{\beta}^{\beta} \right|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right|}, \\ b_{ij}^+ &= \frac{\sum_{\alpha \in I_{r_2, p}\{j\}} \left| (\mathbf{B} \mathbf{B}^*)_{j\cdot} (\mathbf{b}_{i\cdot}^*)_{\alpha}^{\alpha} \right|}{\sum_{\alpha \in I_{r_2, p}} \left| (\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha} \right|}. \end{aligned} \quad (26)$$

Since by Theorem 3.5  $\mathbf{X}_{LS} = \mathbf{A}^+ \mathbf{D} \mathbf{B}^+$ , then an entry of  $\mathbf{X}_{LS} = (x_{ij})$  is

$$x_{ij} = \sum_{s=1}^q \left( \sum_{k=1}^m a_{ik}^+ d_{ks} \right) b_{sj}^+. \quad (27)$$

Denote by  $\hat{\mathbf{d}}_{\cdot s}$  the  $s$ th column of  $\mathbf{A}^* \mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{C}^{n \times q}$  for all  $s = \overline{1, q}$ . It follows from  $\sum_k \mathbf{a}_{\cdot k}^+ d_{ks} = \hat{\mathbf{d}}_{\cdot s}$  that

$$\begin{aligned} \sum_{k=1}^m a_{ik}^+ d_{ks} &= \sum_{k=1}^m \frac{\sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot k}^*)_{\beta}^{\beta} \right|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right|} \cdot d_{ks} = \\ &= \frac{\sum_{\beta \in J_{r_1, n}\{i\}} \sum_{k=1}^m \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot k}^*)_{\beta}^{\beta} \right| \cdot d_{ks}}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right|} = \frac{\sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\hat{\mathbf{d}}_{\cdot s})_{\beta}^{\beta} \right|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right|} \end{aligned} \quad (28)$$

Suppose  $\mathbf{e}_{\cdot s}$  and  $\mathbf{e}_s$  are respectively the unit row-vector and the unit column-vector whose components are 0, except the  $s$ th components, which are 1.

Substituting (28) and (26) in (27), we obtain

$$x_{ij} = \sum_{s=1}^q \frac{\sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} \left( \hat{\mathbf{d}}_{\cdot s} \right)_{\beta}^{\beta} \right|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right|} \frac{\sum_{\alpha \in I_{r_2, p}\{j\}} |(\mathbf{B} \mathbf{B}^*)_{j \cdot} (\mathbf{b}_{\cdot s}^*)_{\alpha}^{\alpha}|}{\sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|}.$$

Since

$$\hat{\mathbf{d}}_{\cdot s} = \sum_{l=1}^n \mathbf{e}_{\cdot l} \hat{d}_{ls}, \quad \mathbf{b}_{\cdot s}^* = \sum_{t=1}^p b_{st}^* \mathbf{e}_t, \quad \sum_{s=1}^q \hat{d}_{ls} b_{st}^* = \tilde{d}_{lt}, \quad (29)$$

then we have

$$\begin{aligned} x_{ij} &= \\ &= \frac{\sum_{s=1}^q \sum_{t=1}^p \sum_{l=1}^n \sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{e}_{\cdot l})_{\beta}^{\beta} \right| \hat{d}_{ls} b_{st}^* \sum_{\alpha \in I_{r_2, p}\{j\}} |(\mathbf{B} \mathbf{B}^*)_{j \cdot} (\mathbf{e}_t)_{\alpha}^{\alpha}|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right| \sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|} = \\ &= \frac{\sum_{t=1}^p \sum_{l=1}^n \sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{e}_{\cdot l})_{\beta}^{\beta} \right| \tilde{d}_{lt} \sum_{\alpha \in I_{r_2, p}\{j\}} |(\mathbf{B} \mathbf{B}^*)_{j \cdot} (\mathbf{e}_t)_{\alpha}^{\alpha}|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right| \sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|}. \end{aligned} \quad (30)$$

Denote by

$$d_{it}^{\mathbf{A}} := \sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} \left( \tilde{\mathbf{d}}_{\cdot t} \right)_{\beta}^{\beta} \right| = \sum_{l=1}^n \sum_{\beta \in J_{r_1, n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{e}_{\cdot l})_{\beta}^{\beta} \right| \tilde{d}_{lt}$$

the  $t$ th component of a row-vector  $\mathbf{d}_{i \cdot}^{\mathbf{A}} = (d_{i1}^{\mathbf{A}}, \dots, d_{ip}^{\mathbf{A}})$  for all  $t = \overline{1, p}$ . Substituting it in (30), we have

$$x_{ij} = \frac{\sum_{t=1}^p d_{it}^{\mathbf{A}} \sum_{\alpha \in I_{r_2, p}\{j\}} |(\mathbf{B} \mathbf{B}^*)_{j \cdot} (\mathbf{e}_t)_{\alpha}^{\alpha}|}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A})_{\beta}^{\beta} \right| \sum_{\alpha \in I_{r_2, p}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|}.$$

Since  $\sum_{t=1}^p d_{it}^{\mathbf{A}} \mathbf{e}_t = \mathbf{d}_{i \cdot}^{\mathbf{A}}$ , then it follows (18).

If we denote by

$$d_{lj}^{\mathbf{B}} := \sum_{t=1}^p \tilde{d}_{lt} \sum_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j\cdot}(\mathbf{e}_{t\cdot})_{\alpha}^{\alpha}| = \sum_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j\cdot}(\tilde{\mathbf{d}}_{l\cdot})_{\alpha}^{\alpha}| \quad (31)$$

the  $l$ th component of a column-vector  $\mathbf{d}_{\cdot j}^{\mathbf{B}} = (d_{1j}^{\mathbf{B}}, \dots, d_{jn}^{\mathbf{B}})^T$  for all  $l = \overline{1, n}$  and substitute it in (30), we obtain

$$x_{ij} = \frac{\sum_{l=1}^n \sum_{\beta \in J_{r_1,n}\{i\}} |(\mathbf{A}^*\mathbf{A})_{\cdot i}(\mathbf{e}_{\cdot l})_{\beta}^{\beta}| d_{lj}^{\mathbf{B}}}{\sum_{\beta \in J_{r_1,n}} |(\mathbf{A}^*\mathbf{A})_{\beta}^{\beta}| \sum_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*)_{\alpha}^{\alpha}|}.$$

Since  $\sum_{l=1}^n \mathbf{e}_{\cdot l} d_{lj}^{\mathbf{B}} = \mathbf{d}_{\cdot j}^{\mathbf{B}}$ , then it follows (17).

iii) If  $\mathbf{A} \in \mathbb{C}^{m \times n}$ ,  $\mathbf{B} \in \mathbb{C}^{p \times q}$  and  $r_1 = n$ ,  $r_2 < p$ , then by Theorem 2.1 and Remark 2.1 the Moore-Penrose inverses  $\mathbf{A}^+ = (a_{ij}^+) \in \mathbb{C}^{n \times m}$  and  $\mathbf{B}^+ = (b_{ij}^+) \in \mathbb{C}^{q \times p}$  possess the following determinantal representations respectively,

$$a_{ij}^+ = \frac{\det(\mathbf{A}^*\mathbf{A})_{\cdot i}(\mathbf{a}_{\cdot j}^*)}{\det(\mathbf{A}^*\mathbf{A})},$$

$$b_{ij}^+ = \frac{\sum_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j\cdot}(\mathbf{b}_{i\cdot}^*)_{\alpha}^{\alpha}|}{\sum_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*)_{\alpha}^{\alpha}|}. \quad (32)$$

Since by Theorem 3.5  $\mathbf{X}_{LS} = \mathbf{A}^+\mathbf{D}\mathbf{B}^+$ , then an entry of  $\mathbf{X}_{LS} = (x_{ij})$  is (27). Denote by  $\hat{\mathbf{d}}_{\cdot s}$  the  $s$ th column of  $\mathbf{A}^*\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{C}^{n \times q}$  for all  $s = \overline{1, q}$ . It follows from  $\sum_k \mathbf{a}_{\cdot k}^* d_{ks} = \hat{\mathbf{d}}_{\cdot s}$  that

$$\sum_{k=1}^m a_{ik}^+ d_{ks} = \sum_{k=1}^m \frac{\det(\mathbf{A}^*\mathbf{A})_{\cdot i}(\mathbf{a}_{\cdot k}^*)}{\det(\mathbf{A}^*\mathbf{A})} \cdot d_{ks} = \frac{\det(\mathbf{A}^*\mathbf{A})_{\cdot i}(\hat{\mathbf{d}}_{\cdot s})}{\det(\mathbf{A}^*\mathbf{A})} \quad (33)$$

Substituting (33) and (32) in (27), and using (29) we have

$$x_{ij} = \sum_{s=1}^q \frac{\det(\mathbf{A}^*\mathbf{A})_{\cdot i}(\hat{\mathbf{d}}_{\cdot s})}{\det(\mathbf{A}^*\mathbf{A})} \frac{\sum_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j\cdot}(\mathbf{b}_{s\cdot}^*)_{\alpha}^{\alpha}|}{\sum_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*)_{\alpha}^{\alpha}|} =$$

$$\begin{aligned}
& \frac{\sum_{s=1}^q \sum_{t=1}^p \sum_{l=1}^n \det(\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{e}_{.l}) \hat{d}_{ls} b_{st}^* \sum_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j.}(\mathbf{e}_{t.})_{\alpha}^{\alpha}|}{\det(\mathbf{A}^* \mathbf{A}) \sum_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*)_{\alpha}^{\alpha}|} = \\
& \frac{\sum_{t=1}^p \sum_{l=1}^n \det(\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{e}_{.l}) \tilde{d}_{lt} \sum_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j.}(\mathbf{e}_{t.})_{\alpha}^{\alpha}|}{\det(\mathbf{A}^* \mathbf{A}) \sum_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*)_{\alpha}^{\alpha}|}. \tag{34}
\end{aligned}$$

If we integrate (31) in (34), then we get

$$x_{ij} = \frac{\sum_{l=1}^n \det(\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{e}_{.l}) d_{lj}^{\mathbf{B}}}{\det(\mathbf{A}^* \mathbf{A}) \sum_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*)_{\alpha}^{\alpha}|}.$$

Since again  $\sum_{l=1}^n \mathbf{e}_{.l} d_{lj}^{\mathbf{B}} = \mathbf{d}_{.j}^{\mathbf{B}}$ , then it follows (21), where  $\mathbf{d}_{.j}^{\mathbf{B}}$  is (19).

If we denote by

$$d_{it}^{\mathbf{A}} := \sum_{l=1}^n \det(\mathbf{A}^* \mathbf{A})_{.i}(\tilde{\mathbf{d}}_{.t}) = \sum_{l=1}^n \det(\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{e}_{.l}) \tilde{d}_{lt}$$

the  $t$ th component of a row-vector  $\mathbf{d}_{i.}^{\mathbf{A}} = (d_{i1}^{\mathbf{A}}, \dots, d_{ip}^{\mathbf{A}})$  for all  $t = \overline{1, p}$  and substitute it in (34), we obtain

$$x_{ij} = \frac{\sum_{t=1}^p d_{it}^{\mathbf{A}} \sum_{\alpha \in I_{r_2,p}\{j\}} |(\mathbf{B}\mathbf{B}^*)_{j.}(\mathbf{e}_{t.})_{\alpha}^{\alpha}|}{\det(\mathbf{A}^* \mathbf{A}) \sum_{\alpha \in I_{r_2,p}} |(\mathbf{B}\mathbf{B}^*)_{\alpha}^{\alpha}|}.$$

Since again  $\sum_{t=1}^p d_{it}^{\mathbf{A}} \mathbf{e}_{t.} = \mathbf{d}_{i.}^{\mathbf{A}}$ , then it follows (22), where  $\mathbf{d}_{i.}^{\mathbf{A}}$  is (16).

iii) The proof is similar to the proof of iii). ■

## 4 An example

In this section, we give an example to illustrate our results. Let us consider the matrix equation

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{D}, \tag{35}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & i & i \\ i & -1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -i \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} i & 1 & -i \\ -1 & i & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \\ 0 & 1 & i \end{pmatrix}.$$

Since  $\text{rank } \mathbf{A} = 2$  and  $\text{rank } \mathbf{B} = 1$ , then we have the case (ii) of Theorem 3.6. We shall find the least squares solution of (35) by (17). Then we have

$$\mathbf{A}^* \mathbf{A} = \begin{pmatrix} 3 & 2i & 3i \\ -2i & 3 & 2 \\ -3i & 2 & 3 \end{pmatrix}, \quad \mathbf{B} \mathbf{B}^* = \begin{pmatrix} 3 & -3i \\ 3i & 3 \end{pmatrix}, \quad \tilde{\mathbf{D}} = \mathbf{A}^* \mathbf{D} \mathbf{B}^* = \begin{pmatrix} 1 & -i \\ -i & -1 \\ -i & -1 \end{pmatrix},$$

$$\text{and } \sum_{\alpha \in I_{1,2}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}| = 3 + 3 = 6,$$

$$\sum_{\beta \in J_{2,3}} |(\mathbf{A}^* \mathbf{A})_{\beta}^{\beta}| = \det \begin{pmatrix} 3 & 2i \\ -2i & 3 \end{pmatrix} + \det \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} + \det \begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix} = 12.$$

By (15), we can get

$$\mathbf{d}_{.1}^{\mathbf{B}} = \begin{pmatrix} 1 \\ -i \\ -i \end{pmatrix}, \quad \mathbf{d}_2^{\mathbf{B}} = \begin{pmatrix} -i \\ -1 \\ -1 \end{pmatrix}.$$

Since  $(\mathbf{A}^* \mathbf{A})_{.1} (\mathbf{d}_{.1}^{\mathbf{B}}) = \begin{pmatrix} 1 & 2i & 3i \\ -i & 3 & 2 \\ -i & 2 & 3 \end{pmatrix}$ , then finally we obtain

$$x_{11} = \frac{\sum_{\beta \in J_{2,3}\{i\}} |(\mathbf{A}^* \mathbf{A})_{.1} (\mathbf{d}_{.1}^{\mathbf{B}})_{\beta}^{\beta}|}{\sum_{\beta \in J_{2,3}} |(\mathbf{A}^* \mathbf{A})_{\beta}^{\beta}| \sum_{\alpha \in I_{1,2}} |(\mathbf{B} \mathbf{B}^*)_{\alpha}^{\alpha}|} = \frac{\det \begin{pmatrix} 1 & 2i \\ -i & 3 \end{pmatrix} + \det \begin{pmatrix} 1 & 3i \\ -i & 3 \end{pmatrix}}{72} = -\frac{1}{72}.$$

Similarly,

$$x_{12} = \frac{\det \begin{pmatrix} -i & 2i \\ -1 & 3 \end{pmatrix} + \det \begin{pmatrix} -i & 3i \\ -1 & 3 \end{pmatrix}}{72} = -\frac{i}{72},$$

$$\begin{aligned}
x_{21} &= \frac{\det \begin{pmatrix} 3 & 1 \\ -2i & -i \end{pmatrix} + \det \begin{pmatrix} -i & 2 \\ -i & 3 \end{pmatrix}}{72} = -\frac{2i}{72}, \\
x_{22} &= \frac{\det \begin{pmatrix} 3 & -i \\ -2i & -1 \end{pmatrix} + \det \begin{pmatrix} -1 & 2 \\ -1 & 3 \end{pmatrix}}{72} = -\frac{2}{72}, \\
x_{31} &= \frac{\det \begin{pmatrix} 3 & 1 \\ -3i & -i \end{pmatrix} + \det \begin{pmatrix} 3 & -i \\ 2 & -i \end{pmatrix}}{72} = -\frac{i}{72}, \\
x_{32} &= \frac{\det \begin{pmatrix} 3 & -i \\ -3i & -1 \end{pmatrix} + \det \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix}}{72} = -\frac{1}{72}.
\end{aligned}$$

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